Modeling the Spread of Frostbite

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Problem Statement

Frostbite is a medical condition characterized by a freezing and ultimate death of tissue at subzero skin interface temperatures[1]. The body is unable to supply enough blood to warm the tissue which allows ice crystals to form in and around the cells, with the extremities most susceptible to injury. Most cases are found in groups with high risk to prolonged cold exposure: homeless individuals, winter sports enthusiasts, and mountaineers[1].

Frostbite progresses through three pathophysiological phases – restriction, dilation, and extreme restriction. The normal cutaneous flow is 200-250 mL/min and decreases as a function of decreasing temperature. At a tissue temperature of 15°C, the maximal vascorestriction is reached and cutaneous flow decreases to 20-50 mL/min[2]. Between 15°C and 0°C, the skin enters the dilation phase and the vascorestriction is interrupted by periodic vasodilation with the body pulsing in new blood in an effort to save cold tissue[2]. Below 0°C, the skin enters the extreme restriction phase and the cutaneous blood flow drops to negligible amounts[2]. The goal of this work is to develop a model that describes the dynamic tissue temperature after extreme restriction.

Background and Model

The skin is a complex mosaic of various cell types and extracellular matrix components[3]. For the purpose of this analysis, the skin was simplified into a 5-layer system that was symmetric around the center, similar to the geometry found in the upper arm. The layers were defined as follows: epidermis (in contact with the external air), dermis, subcutaneous fat, deep muscle tissue, and bone (Fig 1). Each layer's thickness and thermal conductivity are outlined in Table 1[4,5].

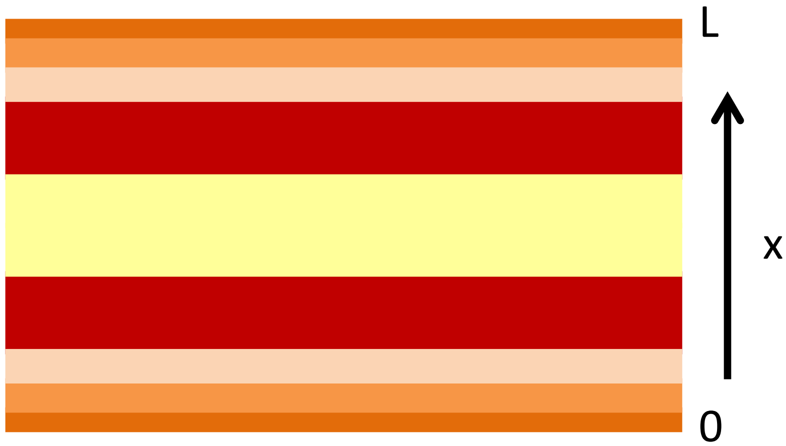


Figure 1. Schematic of simplified skin tissue. Epidermis, dermis, subcutaneous, deep tissue, and bone layers are depicted in dark orange, pale orange, pink, red, and pale yellow, respectively.

Table 1. Tissue Properties (Male, Upper arm)[4]

Tissue	Thickness (cm)	Heat Diffusivity (cm^2/s)
Epidermis	0.08	5.57e-4
Dermis	0.2	1.14e-3
Subcutaneous (Fat)	1.0	5.43e-4
Deep Tissue (Muscle)	3.0	1.25e-3
Bone[5]	1.2	0.11e-1

These layers were chosen to match the typical clinical grading of frostbite cases. First degree frostbite, or frostnip, and second degree frostbite are the freezing of the epidermis and dermis, respectively. These grades of frostbite are able to be repaired by the body. Third and fourth degree frostbite are the freezing of the subcutaneous and deep tissue layers, respectively, and result in permanent damage of tissue and ultimately necrosis. Therefore, it was important to distinguish each layer as a separate entity in order to accurately model the kinetics of frostbite.

During the extreme restriction phase, blood flow is negligible and no new heat is added into the tissue. Therefore, conduction is the sole driving force for heat loss and allowed for the use of the simplified heat equation (1) to model the dynamic tissue temperature.

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} \tag{1}$$

The following simplifying assumptions were also made:

1. The tissue layers are uniform
2. Each tissue has its own thermal diffusivity and heat transfer coefficient
3. No additional convection or heat generation occur in the body
4. No convection occurs at the air-skin interface (i.e. no wind)

Finally, the following initial and boundary conditions were assumed:

$$I.C.: \quad T(0,x) = 37 \;\; and \; B.C.: T(t,0) = T(t,L) = T_{air}$$

Analytical Solution

In order to obtain the analytical solution, the model was further simplified in order to make it tractable; namely, the tissue was assumed to be homogeneous throughout. This was achieved by calculating the weighted average of the heat diffusivity throughout the tissue, which was found to be 3.35e-3 cm^2/s. Using this value for k, Equation 1 was solved using separation of variables, where the position- and time-dependent functions were solved separately to obtain the eigenvalues and eigenfunctions for the complete solution. As a final note, the coefficient was obtained by applying the initial conditions and taking the Fourier sine series of the resulting expression. The analytical solution is found in Equation 2 below; the complete process of solving for the solution can be found in Appendix A.

$$T(x,t) = T_{air} + \sum_{n=1}^{\infty} \frac{2*\left(T_{body} - T_{air}\right)*\left(1-(-1)^{n}\right)}{n\Pi} \sin\left(\frac{n\Pi x}{L}\right) e^{-k\left(\frac{n\Pi}{L}\right)^{2} t} \qquad (2)$$

This solution was then implemented in Matlab, taking only the first 100 terms of the infinite sum for temperatures of 0^oC (the minimum temperature required for tissue to reach 0^oC in our model) and -89.2^oC (the coldest recorded temperature ever[6]). The resulting surface plot for

0^{o}C can be found in Figure 2, though it is more instructive to look at 2D plots of the heat distribution, which can be found in Figure 3.

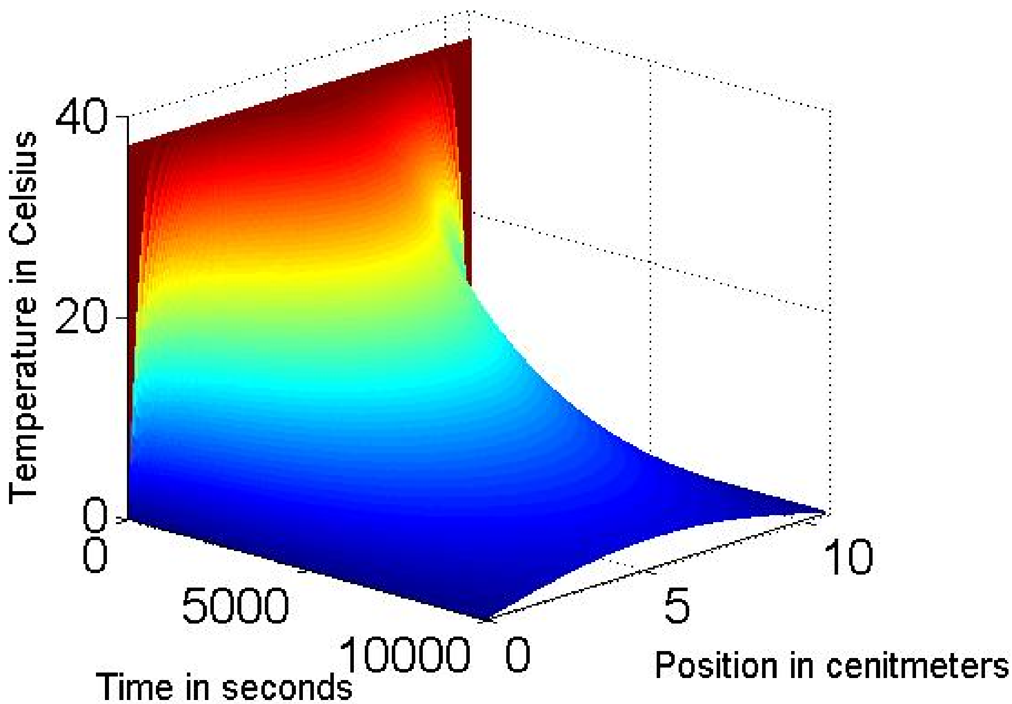


Figure 2: The analytical temperature distribution across the tissue over time for a homogonous tissue model and an external temperature of 0° C.

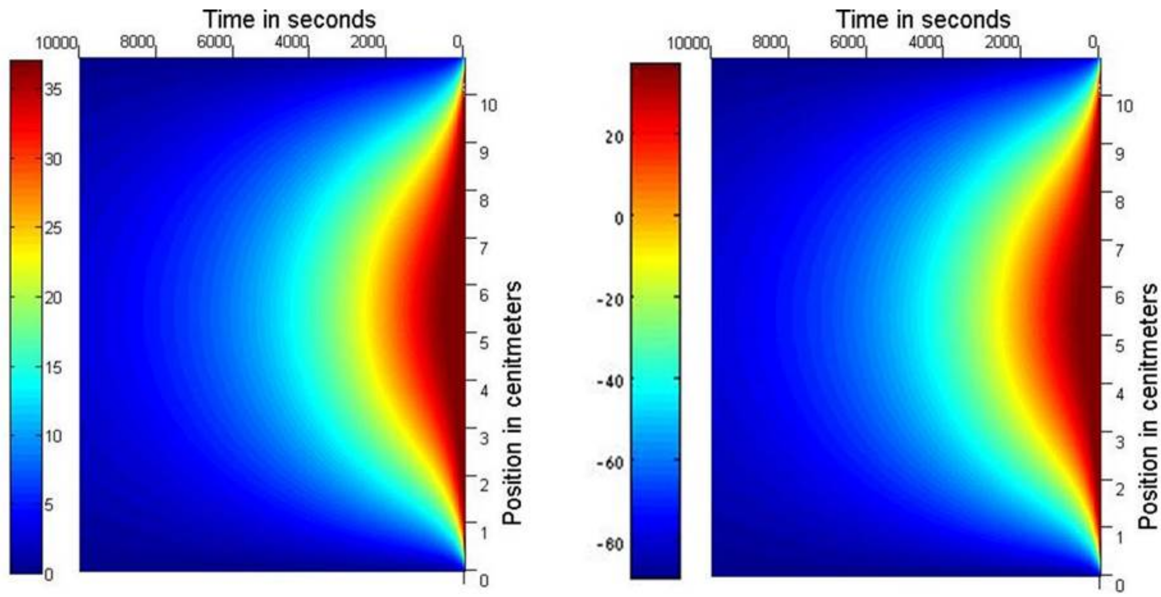


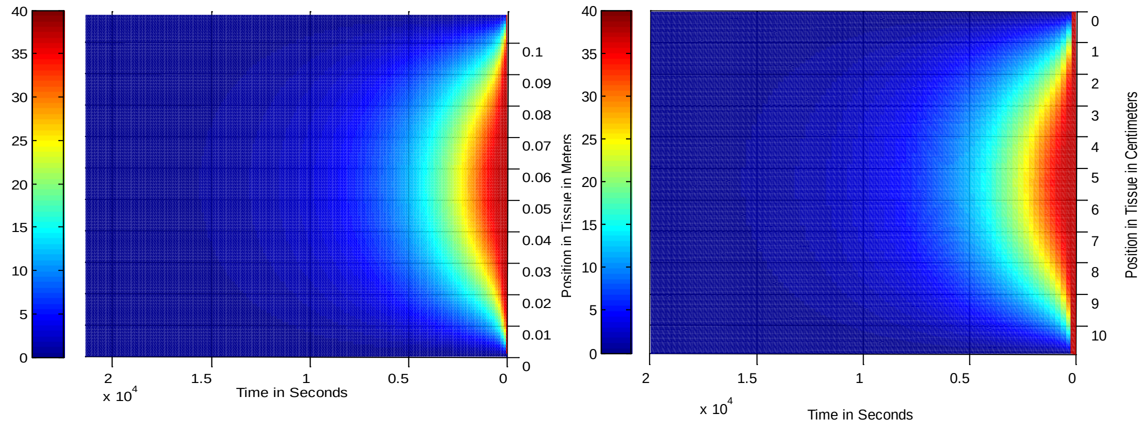
Figure 3: Comparison of analytical result at T_{air} = 0°C (left) and T_{air} = -89.2°C (right), showing that cooling trend does not change, but relative temperatures do (note the scale bars, where on the right 0°C is reached throughout the tissue in approximately 8000s on the left, while it only takes about 2000s on the right).

As expected, the heat distribution in the tissue decreases evenly with time in a half-sine wave pattern. This is an acceptable if rough approximation of the heat loss that occurs during frostbite, where deeper tissues retain heat and avoid permanent damage longer than superficial tissues. From our analytical result, we can confirm that the model is appropriate for the given problem in that the pattern of heat loss approximates that found in vivo, and that by introducing more complexity into the model and implementing numerical methods we may use Equation 1 in deriving a more accurate model of the progression of frostbite in human tissue.

Numerical analysis

Because the analytical solution limits the ability to manipulate variables such as k, a numerical solution was used to visualize various adjustments to our model. All numerical solutions were produced using explicit finite difference.

Our first finite difference model utilized the same parameters as our analytical solution to verify the results of both models. The external temperature was kept at 0° C and a homogenized value for k was applied. The results at these conditions are illustrated in Figure



(a) (b)

Figure 4: The temperature distribution across the tissue and time for a homogonous tissue model and an external temperature of 0° C. (a) is the finite difference solution while (b) is the analytical solution. The color bar indicates the temperature (° C) of the tissue.

We can see that both qualitatively and quantitatively the analytical and numerical solutions agree. It would be useful to obtain the time at which the temperature of the skin falls to 0°, the temperature at which we have stated frostbite sets in. However, because of the methodology of finite difference, the temperature in the tissue will never fall below or reach the temperature of the air surrounding; it will only approach that value as time approaches infinity. In this first case, and in any case where the external temperature is 0° C, the values for tissue temperature never reach zero and therefore no time point can be obtain. If a threshold value was established for which any temperature below that value was considered zero, the time to 0° C could be establish but would be highly variable based on the threshold chosen.

The second finite difference model that was employed divided the tissues up into distinct layers, each with their own thermal properties. The layers and their properties are given in Table 1. The results of numerical analysis with the variegated tissue properties are given in Figure 5.

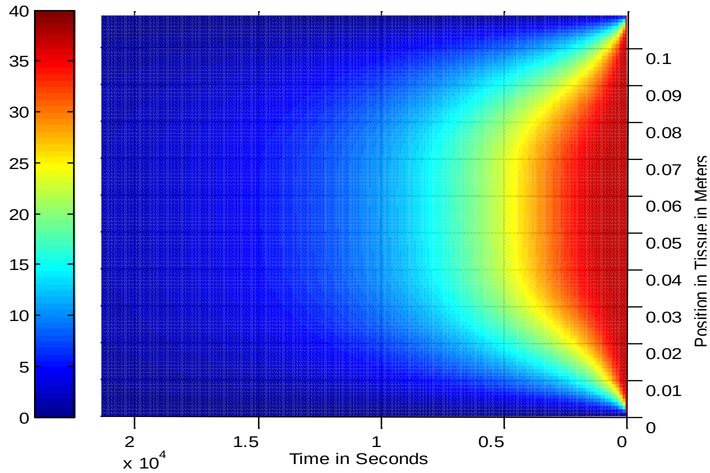


Figure 5. The temperature distribution across the tissue and time for a layered model and an external temperature of 0° C. The color bar indicates the temperature (° C) of the tissue.

As expected, we get a thermal profile quite different from that in the case of homogenous tissue properties. Most strikingly, we notice the region of relative flatness in the central region of the limb cross section; this region represents the bone. The relatively high thermal diffusivity of bone results in an area of dramatically different curvature than the rest of the tissue. The variegation of tissue properties improves the accuracy of the model. Again the time to 0° C cannot be obtained due to the limitation of the numerical method as stated above.

Our third, fourth, fifth, and sixth models look at the effect of external temperature on the temperature distribution in the tissue. Three cases are proposed. The first models a relatively common outdoor winter temperature for regions with relatively cool climates. -5° C was the external temperature for this scenario. The second model was an outdoor extreme temperature of -40° C. The final outdoor temperature scenario considered was the coldest temperature recorded in history, -89.2° C. Our final temperature consideration modeled the tissue being exposed to liquid nitrogen at -196° C. The data for these models is given below in Figure 6 and Table 2.

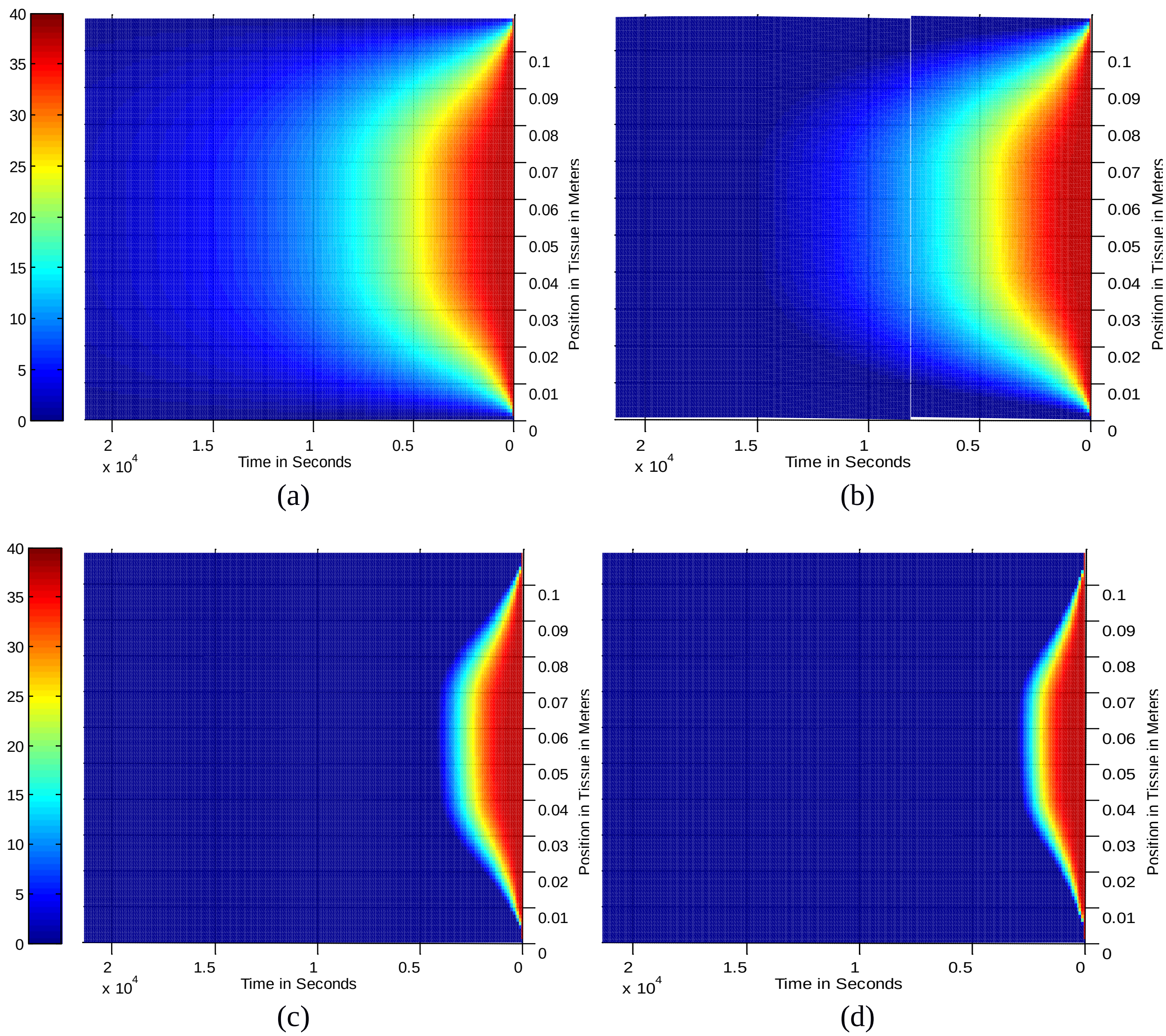


Figure 6: The temperature distribution across the tissue and time for a layered model and an external temperature of (a) -5° C, (b) -40° C, (c) -89.2° C, (d) -196° C. The color bar indicates the temperature (° C) of the tissue.

Table 2: The times at which various tissues freeze in each external condition.

External Temperature (° C)	Time for Dermis to Freeze (s)	Time for Subcutaneous to Freeze (s)	Time to Freeze to Bone (s)
-5	391	9783	15497
-40	10	1616	6068
-89.2	4	834	4059
-196	1.8	496	2836

At lower external temperatures, the temperature profiles demonstrate much sharper gradients and the tissue layers reach 0° C more rapidly. This result is what is intuitively expected in colder temperatures.

Conclusion

Both the analytical and numerical methods used in this analysis produced similar solutions to the model proposed. The numerical solution allowed for a higher degree of manipulation and therefore allowed for a more realistic result. While the qualitative and order of magnitude results produced herein may provide information on the process of frostbite, the quantitative results are highly inaccurate. Several significant contributors to tissue temperature were neglected, including wind chill (convective effects), metabolism (heat generation), and the dynamics of thermal properties (changing k values and freezing energy). Ignoring blood flow and metabolism are not unreasonable assumptions in freezing conditions, however, the neglect of wind chill, freezing energy, and insulation effects of frozen exterior tissues are significant flaws in the given model. Accounting for these effects will make the model more complicated but also will provide results that more closely mimic the actual temperature distribution.

Other than the implementation of convective terms on the surfaces and heat generation within the tissue, a simple manner in which the model could be improved relatively easily is the implementation of cylindrical co-ordinates, which would better approximate the cylindrical shape of the upper arm. Another simple modification that could prove valuable would be the implementation of parameters that better approximate the tissue structure of a toe or finger, as these are the two extremities of the human body that are most commonly affected by frostbite[1].

References

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4 Cohen, M. L. Measurement of the thermal properties of human skin. A review. *Journal of investigative dermatology* **69**, 333-338 (1977).

5 Biyikli, S., Modest, M. F. & Tarr, R. Measurements of thermal properties for human femora. *Journal of biomedical materials research* **20**, 1335-1345 (1986).

6 Wikipedia. *Lowest Temperature Recorded on Earth*, <http://en.wikipedia.org/wiki/Lowest_temperature_recorded_on_Earth#cite_note-1> (2013).

Appendix A: Solution to Analytic Equation

Analytical Solution

① $\frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial x^2}$ Let $T(x,t) = X(x) \cdot Q(t)$ ②

Sub ② into ① $\rightarrow$ $\frac{Q'(t)}{KQ(t)} = \frac{X''(x)}{X(x)} = -\lambda$ because each is completely independent.

In Space: If $\lambda = 0$

$\frac{d^2X}{dx^2} + \lambda X = 0$ $X(x) = C_1 + C_2 x \Rightarrow$ BC's: $X(0) = 0 = C_1$

$X(L) = 0 = C_2 L$ $C_1 = C_2 = 0$

Trivial Solution

If $\lambda < 0$

$X(x) = C_1 \cosh(\sqrt{-\lambda}\, x) + C_2 \sinh(\sqrt{-\lambda}\, x) \Rightarrow$ BC's:

$X(0) = 0 = C_1$

$X(L) = C_2 \sinh(-\sqrt{\lambda}\, L)$

$\Rightarrow C_2 = 0$

$C_1 = C_2 = 0$

Trivial Solution

If $\lambda > 0$

$X(x) = C_1 \cos(x\sqrt{\lambda}) + C_2 \sin(x\sqrt{\lambda})$ BC's:

$X(0) = 0 = C_1$

$X(L) = 0 = C_2 \sin(L\sqrt{\lambda})$

So $\lambda = \left(\frac{n\pi}{L}\right)^2$ for $n = 1, 2, 3 \ldots$ are eigenvalues

and $X(x) = C_2 \sin\left(\frac{n\pi x}{L}\right)$ for $n = 1, 2, 3 \ldots$ are eigenfunctions

In Time:

$\frac{dQ}{dt} + \lambda K Q = 0$ so $\int \frac{1}{Q}\, dQ = \int -\lambda K\, dt$

$\ln Q = -\lambda K t + C$

$Q(t) = Ce^{-\lambda K t}$

Recombining Terms $\Rightarrow Q(t) \cdot X(x)$

$$\Rightarrow T(x,t) = B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

* B_n contains constants from both time and space terms.

Using I.C. of $T(x,0) = T_{body}$ And Fourier Sine Series

$$B_n = \frac{2}{L}\int_0^L T_{body} \sin\left(\frac{n\pi x}{L}\right) dx$$

$$B_n = \frac{2T_{body}}{L}\left[\frac{-L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L\right]$$

$$B_n = \frac{2T_{body}}{n\pi}\left[-\cos(n\pi) + 1\right]$$

So for odd n, $B_n = \frac{4T_{body}}{n\pi}$

even n, $B_n = 0$

Final Solution:

$$T(x,t) = \sum_{n=1}^{\infty} \frac{2T_{body}\left(1-(-1)^n\right)}{n\pi} \sin\left(\frac{n\pi x}{L}\right) e^{-K\left(\frac{n\pi}{L}\right)^2 t}$$

For BC $T(0,t) = T(L,t) = T_{air}$:

$$T(x,t) = T_{air} + \sum_{n=1}^{\infty} \frac{2T_{body}\left(1-(-1)^n\right)}{n\pi} \sin\left(\frac{n\pi x}{L}\right) e^{-K\left(\frac{n\pi}{L}\right)^2 t}$$

Appendix B: Matlab Code for Analytic Solution

```
clear

%PDE: dT/dt = k d^2T/dx^2

%Parameters:
L = 5.45*2; %cm
k = 3.35e-3; %cm^2/s
Tbody0 = 37; %degC
Tair = -89.2; %degC

dx = .02;
dt = .1;
tf = 10000;

xmesh = 0:dx:L;
xlength = length(xmesh);
tmesh = 0:dt:tf;
tlength = length(tmesh);

%IC's
heat = zeros((tf/dt+1),(L/dx));
heat(1,:) = Tbody0;

x=0;
t=0;
bnew=0;%zeros(2,1);
Tbody = Tbody0;
heat1 = 0;

%BC's
heat(:, 1) = Tair;
heat(:, xlength) = Tair;

%Analytical Solution as a surface plot --> First 5 terms in infinte series

for t = 2:tlength
     for x = 2:xlength-1
          for n = 1:2:101
            b(n) = (4*(Tbody-Tair)/(n*pi))* ...
                sin(n*pi*(x*dx)/L)*exp(-k*((n*pi/L)^2)*(t*dt));
          end
          heat(t,x) = sum(b)+Tair;
      end
end

% surf(tmesh(1:250:end),xmesh,heat(1:250:end,:)', 'edgecolor', 'none')
% set(gca,'FontSize',20)
% axis([0 10000 0 10.75 -90 40])
% xlabel('Time in seconds','FontSize',15),ylabel('Position in
cenitmeters','FontSize',15)...
%     ,zlabel('Temperature in Celsius','FontSize',15)
```

Appendix C: Matlab Code for Numeric Solution

```
%BENG 221 Project
%Numerical Solution

%Clear and Format
clc
clear

%Set Parameters
L=1; %cm
k=5.57E-8; %cm^2/s
T1=37; %Deg C, Body Temperature
T2=-196; %Deg C, Outside temperature

%Thickness of each layer (m)
epi=80E-6; %Epidermis
der=0.002; %Dermis
sub=0.01; %Subcutaneous
dee=0.03; %Deep Tissue
bon=.025; %Bone

%Define layers for mesh
th1= epi;
th2= th1+der;
th3= th2+sub;
th4= th3+dee;
th5= th4+bon;
th6= th5+dee;
th7= th6+sub;
th8= th7+der;
th9= th8+epi;

% domain
dx = 0.001; % step size in x dimension
dt = 0.1; % step size in t dimension
xmesh = 0:dx:th9; % domain in x
tmesh = 0:dt:21400; % domain in t

%%%solution using finite differences

%Set up solutoion Mesh
nx = length(xmesh); % number of points in x dimension
nt = length(tmesh); % number of points in t dimension
sol_fd = zeros(nt,nx);

%Set Initial Conditions
sol_fd(:,nx) = T2;
sol_fd(1,2:nx-1) = T1;% initial conditions; delta impulse at center
sol_fd(:,1) = T2;

%Solve
for t = 1:nt-1
    for x = 2:nx-1
```

```
        pos=x*dx;
        if pos<th1
            k=5.57E-8; %thermal difusivity of epidermis (m^2/s)
        elseif pos<=th2 && pos>th1
            k=1.14E-7; %thermal difusivity of dermis (m^2/s)
        elseif pos<=th3 && pos>th2
            k=5.43E-8; %thermal difusivity of subcutaneous (m^2/s)
        elseif pos<=th4 && pos>th3
            k=1.25E-7; %thermal difusivity of epidermis (m^2/s)
        elseif pos<=th5 && pos>th4
            k=0.11E-5; %thermal difusivity of bone (m^2/s)
        elseif pos<=th6 && pos>th5
            k=1.25E-7; %thermal difusivity of epidermis (m^2/s)
        elseif pos<=th7 && pos>th6
            k=5.43E-8; %thermal difusivity of subcutaneous (m^2/s)
        elseif pos<=th8 && pos>th7
            k=1.14E-7; %thermal difusivity of dermis (m^2/s)
        else
            k=5.57E-8; %thermal difusivity of epidermis (m^2/s)
        end
        stepsize = k * dt / dx^2; % stepsize for numerical integration
        %Calculation of new values
        sol_fd(t+1,x) = sol_fd(t,x) + stepsize *(sol_fd(t,x+1)-2 *
sol_fd(t,x) + sol_fd(t,x-1));
    end
end

%Calculate time for freezing

indder=find(sol_fd(:,round(th2/dx))<=0,1,'first');
tderfr=dt*indder(1,1)
indsub=find(sol_fd(:,round(th3/dx))<=0,1,'first');
tsubfr=dt*indsub(1,1)
inddee=find(sol_fd(:,round(th4/dx))<=0,1,'first');
tsdeefr=dt*inddee(1,1)



% %Plot
% clf
% figure(1)
% surf(xmesh,tmesh(1:1000:nt),sol_fd(1:1000:nt,:), 'edgecolor', 'none')
% axis([0 nx*dx 0 nt*dt 0 40 0 40])
% xlabel('Position in Tissue in Meters', 'rot', 90)
% ylabel('Time in Seconds')
% zlabel('Temperature in Celcius')
% colorbar('location', 'WestOutside')
```